Last update: 6:25pm, Mar 22.

Math 565: Functional Analysis HOMEWORK 5 **Due: Mar 24, 23:59**

1. Pick **one** of the topological vector spaces below and prove that it is complete.

(a) $L^1_{\mathrm{loc}}(\mathbb{R}^d, \lambda)$ with the topology generated by the seminorms $p_n(f) \coloneqq \|f \cdot \mathbb{1}_{B_n}\|_1$, where B_n is the open ball of radius n about 0, for $n \in \mathbb{N}$.

(b) $L^0(X, \mu)$, where (X, μ) is a finite measure space, with the topology of convergence in measure, i.e. generated by the quasi-seminorms $p_n(f) \coloneqq \mu(|f| > 1/n)$, where $n \in \mathbb{N}^+$.

(c) $C(X)$, where X is a locally compact Hausdorff topological space, with the compact-open topology, equivalently, the topology generated by the seminorms $p_K(f) \coloneqq \sup_{x \in K} |f(x)|$, where $K \subseteq X$ is compact.

(d) $C^\infty([0,1])$ with the topology generated by the seminorms $p_n(f) \coloneqq \|f^{(n)}\|_u$, where $n \in \mathbb{N}$.

2. Let X be a normed vector space and $D \subseteq X$ be a dense subset. For each $r > 0$, let $\overline{B}^*_r \coloneqq \{f \in X^* : \|f\| \leqslant r\}$, i.e. the closed norm ball of radius r in X^*. Prove that $\overline{B}^*_r$ equipped with the weak* topology is homeomorphic to $\mathrm{Lin}(D_1, \overline{B}^{\mathbb{C}}_r)$, where $D_1 \coloneqq D \cap B^X_1$. In particular, if X is separable, then $\overline{B}^*_r$ is compact and metrizable, hence sequentially compact.

HINT: You can prove the homeomorphism using nets, as we did in class when proving that $\overline{B}^*_r$ is homeomorphic to $\mathrm{Lin}(\overline{B}^X_1, \overline{B}^{\mathbb{C}}_r)$.

3. Let X be an infinite-dimensional Banach space. Prove:

(a) Every weak*-Cauchy sequence $(f_n)_{n \in \mathbb{N}} \subseteq X^*$ weak*-converges in X^*. Furthermore, $(f_n)_{n \in \mathbb{N}}$ is norm-bounded.

(b) [1] [*Optional*] Prove that despite part (a), the weak* topology on X^* is not complete, in fact, $\mathrm{Lin}(X, \mathbb{C})$ is its weak*-completion, i.e. $\mathrm{Lin}(X, \mathbb{C})$ is weak*-complete and X^* is weak*-dense in it.

(c) Every weak*-open set $U \subseteq X^*$ contains a coset of a subspace of finite codimension[2]. In particular, U is norm-unbounded.

(d) Every norm-bounded subset of X^* is weak*-nowhere dense.

(e) X^* is weak*-meager. In particular, the weak* topology on X^* is not metrizable.

HINT: The Birkhoff–Kakutani theorem implies that if X^* is metrizable then it admits a translation-invariant compatible metric.

REMARK: Contrast the non-metrizability of the weak* topology on X^* with the metrizability of its norm unit ball when X is separable (Question 2).

[1] Thanks Zhaoshen Zhai for pointing out that unlike *sequences*, convergent *nets* may be unbounded, and for providing the correct statement about the completion of X^*.

[2] The **codimension** of a subspace W of a vector space V is the dimension of the quotient V/W.

4. Follow the steps below to prove that for a Banach space X, the subspace $\widehat{X}$ is weak* dense in X^{**}; in fact, the norm-1 open ball of $\widehat{X}$ is weak* dense in the norm-1 open ball of X^{**}.

Let $F \in X^{**}$ and fix a basic weak*-open neighbourhood of F, i.e. a set of the form

$$[f_1, f_2, \ldots, f_n; \varepsilon] := \{F' \in X^{**} : |F'(f_i) - F(f_i)| < \varepsilon \text{ for all } i \leqslant n\},$$

where $f_1, f_2, \ldots, f_n \in X^*$ and $\varepsilon > 0$.

(a) It is enough to show that for each $\delta > 0$ there is $x \in X$ of norm $< \|F\| + \delta$ such that $\widehat{x} \in [f_1, f_2, \ldots, f_n; \varepsilon]$; in fact, we will find an $x \in X$ of norm $< \|F\| + \delta$ such that $\widehat{x}(f_i) = F(f_i)$ for all $i \leqslant n$.

(b) Let $K := \ker T$ where $T : X \to \mathbb{C}^n$ is defined by $x \mapsto (f_1(x), f_2(x), \ldots, f_n(x))$. Then for each $i \leqslant n$, the functional f_i factors through the quotient map $\pi : X \to X/K$, yielding a continuous linear functional $\widetilde{f_i} : X/K \to \mathbb{C}$ given by $x + K \mapsto f_i(x)$. The linear span Y of $\left\{\widetilde{f_1}, \widetilde{f_2}, \ldots, \widetilde{f_n}\right\}$ is equal to $(X/K)^*$. Hence $Y^* = (X/K)^{**} = \widehat{(X/K)}$

(c) Since $F|_Y \in Y^*$, there is a coset $C \in X/K$ such that $F|_Y = \widehat{C}$. We can choose a representative $x \in C$ such that $\|x\| < \|F\| + \delta$.

5. Prove that for a finite-dimensional normed vector space X, the weak topology on X coincides with the norm topology. Deduce that the weak* topology on X^* coincides with the norm topology on X^*.

6. [3] Follow the steps below to give an alternative (much simpler) proof that closed subspaces of reflexive Banach spaces are reflexive.

Let X be a reflexive normed vector space (hence a Banach space), and let $Y \subseteq X$ be a closed subspace.

(a) Y^{**} embeds into X^{**} via the double-transpose ι^{**} of the inclusion map $\iota : Y \hookrightarrow X$. Explicitly, $\iota^{**}(F)(g) = F(g|_Y)$ for all $F \in Y^{**}$ and $g \in X^*$.

(b) For every $F \in Y^{**}$, we have $\iota^{**}(F) = \widehat{x}$ for some $x \in X$, and it suffices to show that $x \in Y$. Supposing that $x \notin Y$, we get some $g \in X^*$ such that $\widehat{x}(g) \neq 0$ but $\iota^{**}(F)(g) = 0$, a contradiction.

7. Let X, Y be Banach spaces.

(a) Prove that every weak*-convergent sequence in X^* is norm-bounded.

(b) Deduce that every weakly convergent sequence in a normed vector space X is norm-bounded.

(c) Conclude that every weakly convergent sequence in $B(X, Y)$ is norm-bounded. In particular, every strongly convergent sequence in $B(X, Y)$ is norm-bounded.

8. Let H be an inner product space.

[3]Thanks Olivier Lefèvre for suggesting this simpler proof.

(a) Prove the **polarization identity**: for all $x, y \in H$,

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) + i\frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2) = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2$$

(b) Deduce that if H' is another Hilbert space, then a linear map $T : H \to H'$ preserves the inner product if and only if it preserves the norm (i.e. is an isometry).